THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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THE COMMITTEE ON ARITHMETIC AND ALGEBRA.

The following Report, like its predecessor (*v. Gazette*, p. 167), is preliminary and subject to revision. All criticisms and suggestions should be sent to Mr. A. W. Siddons, Harrow School, Middlesex. [W. J. G.]

REPORT OF THE M.A. COMMITTEE ON ARITHMETIC AND ALGEBRA.

1. The Committee consider that there is considerable danger of the true educational value of Arithmetic and Algebra being seriously impaired by reason of a tendency to sacrifice clear understanding to mere mechanical skill.

2. In view of this the Committee recommend:

(*a*) that *easy vivâ voce* examples should be frequently used in both Arithmetic and Algebra;

(*b*) that great stress should be laid on fundamental principles;

(*c*) that, as far as possible, the rules which a pupil uses should be generalisations from his own experience;

(*d*) that, whenever practicable, Geometry should be employed to illustrate Arithmetic and Algebra, and, in particular, that graphs should be used extensively;

(*e*) that many of the harder rules and heavier types of examples, which examinations alone compel us to retain in a school curriculum, should be postponed.

With these as guiding principles the Committee are led to make the following suggestions:—

ARITHMETIC.

3. That, as a preparation for contracted multiplication, pupils should, in multiplying, from the very first be taught to begin with the highest digit of the multiplier.

4. That first principles should be carefully taught before vulgar fractions are begun, special stress being laid on factors with index notation, and the use of the signs $+, -, \times, \div$, and of brackets.

5. That, as far as possible, prime factors should be used in finding H.C.F., L.C.M., and square and cube roots.

6. That much time should not be spent in teaching complicated vulgar fractions.

7. That, in order to facilitate the teaching of decimals, a scale divided decimally (both in inches and centimetres) should be used for the actual measurement of lines; the scale should be subsequently used in the determination of areas and volumes.

8. That in teaching decimals concrete examples should be taken from the metric system.

9. That approximate work with decimals should be introduced early, and that recurring decimals should be treated in this way, the theory of recurring decimals being postponed to a later stage.

10. That the rule for cube root should be omitted.

11. That those parts only of the English tables of weights and measures which are in general use should be taught; and that long questions in reduction (*e.g.* square miles to square inches) should be avoided.

12. That in money sums the system of decimalising money at sight should be largely used, and that answers should, as a rule, not be required beyond the nearest penny.

13. That, in pass examinations in Arithmetic, the use of algebraic symbols should not be prohibited.

14. That rough checks should be constantly employed in arithmetical work.

15. The Committee consider that the mental training afforded by Arithmetic is largely impaired by the existing tendency of textbooks to classify problems and to establish each type as a separate rule.

ALGEBRA.

16. The Committee suggest that beginners should be taught the use of letters to denote numbers, by substituting letters for numbers in suitable illustrations (involving integers) of the fundamental laws underlying Arithmetic and Algebra; they should also be shown how results and processes with which they are familiar in Arithmetic and Geometry can be expressed generally by the use of letters (for example, before beginning Algebra pupils will have been taught how to solve questions on interest and areas without formulae; from these and similar examples they should be led at an early stage of Algebra to construct formulae for themselves).

17. That the fundamental laws of Algebra should be inferred from examples in Arithmetic and Geometry by a rough induction; formal proofs should be postponed.

18. The Committee consider that the use of the minus sign to denote negative quantity may be introduced at the very beginning of Algebra.

19. That, instead of the usual difficult substitutions, practice should be given at an early stage in substituting numbers for letters in identities, and in testing the roots of equations, not necessarily of the first degree, and in plotting graphs of functions.

20. That multiplication by a binomial, and simple addition and subtraction should be done mentally.

21. That the rules for long multiplication and division should be postponed till after problems on x, y equations, and that the method of detached coefficients should be taught, and the analogy with the corresponding processes in Arithmetic should be pointed out.

22. That the methods used in solving equations should be based on and frequently referred back to first principles (the four axioms); and that roots should be tested by substitution.

23. That *graphs* should be introduced as early as possible and be used extensively (certainly in connection with x, y equations).

24. That complicated fractions, hard H.C.F. and L.C.M., and literal equations should be postponed till after easy quadratics.

25. That irrational roots of quadratic equations should be often worked out to a few significant figures.

26. That the algebraic method of proving such theorems as those of Euclid, Book II., should be explained, and that problems similar to Euclid II. 11 should be given among quadratic problems, and that other applications to Geometry which assume a knowledge of Euclid, Books I. and III., should also be given.

27. That in *surds* the chief stress should be laid on numerical evaluation, including rationalisation of denominators.

28. That complicated questions on fractional indices should be avoided.

29. That fractional indices should be illustrated by logarithms taken from four figure tables or from graphs made by the pupils themselves.

30. That work with imaginary quantities should, as far as possible, be avoided in elementary Algebra.

31. That the comparison of the algebraic definition of proportion with that given by Euclid, together with all such phrases as "duplicate ratio," "compounding ratios," etc., and the examples illustrating them, should be dropped out of elementary Algebra.

32. That great stress should be laid on the use of numerical checks in connection with algebraic work.

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33. In view of the great amount of time now required for teaching the various rules connected with our complicated system of weights and measures, this Committee desires to record its unanimous opinion that the interests of education demand the early introduction of a decimal system of weights, measures, and coinage.

TRIPOLAR COORDINATES.

1. The tripolar coordinates of a point P are defined as the squares of its distances from the angular points of the triangle of reference.

The equation of the first degree in x, y, z ($x = PA^2$, etc.) has been considered by E. Lucas (*Mathesis*, 1889) and by de Poulain (*Journal de Mathématiques spéciales*, 1889). I am also indebted to Mr. R. F. Davis for a note in accordance with which a portion of this paper has been rewritten.

In what follows x, y, z will denote the tripolar coordinates; α, β, γ the trilinear coordinates of a point with respect to a triangle of reference ABC; while ξ and η will denote Cartesian coordinates referred to B as origin and BC as the axis of x.

We have at once

$$\left.\begin{aligned} x &= (\xi - c\cos B)^2 + (\eta - c\sin B)^2 \\ y &= \xi^2 + \eta^2 \\ z &= (\xi - a)^2 + \eta^2 \end{aligned}\right\}.$$

On solving this set for ξ and η,

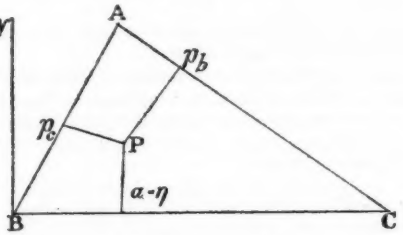
$$\left.\begin{aligned} 2a\xi &= a^2+y-z \\ 4\Delta\eta &= abc\cos A - ax + by\cos C + cz\cos B \end{aligned}\right\}.$$

From the second equation of the last set,

$$\left.\begin{aligned} 4\Delta\alpha &= abc\cos A - ax + by\cos C + cz\cos B \\ 4\Delta\beta &= abc\cos B + ax\cos C - by + cz\cos A \\ 4\Delta\gamma &= abc\cos C + ax\cos B + by\cos A - cz \end{aligned}\right\}.$$

From the cyclic quadrilateral Pp_bAp_c, we have

$$\left.\begin{aligned} x\sin^2 A &= \beta^2+\gamma^2+2\beta\gamma\cos A \\ y\sin^2 B &= \gamma^2+\alpha^2+2\alpha\gamma\cos B \\ z\sin^2 C &= \alpha^2+\beta^2+2\alpha\beta\cos C \end{aligned}\right\}.$$



The identical relation satisfied by x, y, z is obtained from

$$y=\xi^2+\eta^2=(a^2+y-z)^2/4a^2+(abc\cos A-ax+by\cos C+cz\cos B)^2/16\Delta^2,$$

which after some reduction leads to

$$\Sigma a^2x^2-2\Sigma bcyz\cos A-2abc\Sigma ax\cos A+a^2b^2c^2=0.$$

Most of the formulae relating to the straight line and circle are obtained independently of these formulae of transformation. It may however be noticed that while x, y, z are quadratic functions of α, β, γ and of ξ, η, yet their differences $x-y$, etc., are only linear functions, and that in transforming from trilinears or Cartesians to tripolars the degree of the equation remains in general the same, though in exceptional cases it may be lowered.

The Equation of the First Degree.

2. If E is the mean centre of ABC for multiples l, m, n, we have

$$\Sigma(lPA^2)=\Sigma l\,.\,PE^2+\Sigma(lEA)^2).$$

$$\Sigma(l\,.\,EA^2)=\Sigma mna^2/\Sigma l\,;$$

(cf. R. F. Davis, *E. T.*,* vol. LXVII., p. 113)

$$\therefore\ \Sigma lx\equiv lx+my+nz=s$$

represents in general a circle whose centre E has its trilinear coordinates proportional to l/a, m/b, n/c, while its radius ρ is given by

$$\rho^2=PE^2=\{\Sigma lx-\Sigma mna^2/\Sigma l\}/\Sigma l$$
$$=(s\Sigma l-\Sigma mna^2)/(\Sigma l)^2.$$

If Q be another point (x', y', z') and QT be the tangent from Q to this circle,

$$\Sigma lx_1=\Sigma l\,.\,QE^2+\Sigma mnx^2/\Sigma l\,;$$

$$\therefore\ QT^2=QE^2-ET^2=(\Sigma lx_1-s)/\Sigma l.$$

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* *Educational Times, Reprint* (Hodgson).

If the circle cut the sides of ABC at angles θ_1, θ_2, θ_3, we have

$$\cos\theta_1=\frac{2\Delta l}{ap\Sigma l};$$

$$\therefore\ \cos\theta_1:\cos\theta_2:\cos\theta_3=l/a:m/b:n/c.$$

Examples. The equations of some well-known circles are:

(*a*) Circumcircle, $\Sigma ax\cos A=abc.$

(*b*) In-circle, $\Sigma ax=2\Delta(r+2R).$

(*c*) N.P. circle, $\Sigma ax\cos(B-C)=abc(1+2\Pi\cos A).$

(*d*) Brocard circle, $\Sigma ax\cos(A-\omega)=abc\cos\omega.$

(*e*) Any Tucker circle,

$$\Sigma ax\cos(A-\theta)=abc\{\cos^2\theta+\cot\omega\sin\theta\cos\theta+2\sin^2\theta\}/\sin(\omega+\theta).$$

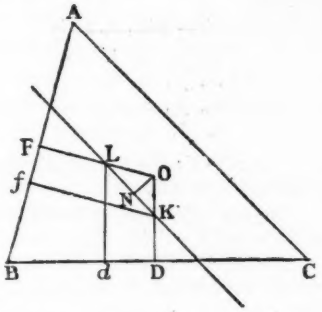
(*f*) The polar circle, $\Sigma ax\cos B\cos C=abc\Pi\cos A.$

3. If $\Sigma l=0$, E is at infinity, and $\Sigma lx=s$ becomes a straight line.

Let now O be the circumcentre; D, E, F the mid-points of the sides, and let

$$\Sigma lx=s;\ \Sigma l=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

cut OD and OF in K and L. Draw ON, Kf, and Kd perpendicular to KL, AB, and BC respectively.



Then OD meets (i) where $y=z$, and $x-y=\frac{s}{l}$, *i.e.* on a parallel to OF through f, where

$$Af^2-fB^2=\frac{s}{l},$$

whence

$$fF=\frac{s}{2cl},$$

and similarly,

$$Dd=\frac{s}{2an};$$

$\therefore$ if θ_1 be the angle at which KL cuts BC,

$$dD/\cos\theta_1=-Ff/\cos(B+\theta_1);$$

$$\therefore\ \cos\theta_1=cl\sin B/\sqrt{c^2l^2+a^2n^2+2acln\cos B}$$

$$=cl\sin B/\sqrt{-\Sigma a^2mn};$$

$$\therefore\ \cos\theta_1:\cos\theta_2:\cos\theta_3=l/a:m/b:n/c.$$

Also

$$ON=OK\cos\theta_1=Ff\cos\theta_1/\sin B$$

$$=\frac{s}{2\sqrt{-a^2mn-b^2nl-c^2lm}}$$

The line $lx+my+nz=lx_1+my_1+nz_1$ is a parallel to (i) through $Q(x_1, y_1, z_1)$, and therefore the perpendicular from (x_1, y_1, z_1) on (i) is

$$(\Sigma lx_1 - s)/2\sqrt{-\Sigma a^2 mn}.$$

The two lines $lx+my+nz=s$, $l'x+m'y+n'z=s'$ will be parallel if

$$l : m : n = l' : m' : n',$$

and perpendicular if $\tan\theta_1 \tan\theta_1' + 1 = 0$,

i.e. if $a^2nn' + ac\cos B(l'n+ln') + c^2ll' = 0$,

or, in the more symmetrical form,

$$a^2(mn'+m'n) + b^2(nl'+l'n) + c^2(lm'+l'm) = 0.$$

Examples.

(i) The equations of OH and OK are

$$\Sigma(b^2-c^2)x = 0 \quad \text{and} \quad \Sigma a^2(b^2-c^2)x = 0.$$

(R. F. Davis, *E. T.*, vol. LXVII., p. 113.)

(ii) Points tripolarly associated (*i.e.* with their tripolar coordinates in the same ratio) are inverse with regard to circumcircle. (Casey, *Sequel*, p. 256.)

(iii) The points $x : y : z = a^2 : b^2 : c^2$ are on the line OH.

(R. F. Davis, *E. T.*, vol. LXVIII., p. 28.)

(iv) The line $lx+my+nz=s$ corresponds to the line whose trilinear equation is $\Sigma bcl(c\beta+b\gamma) = s\Sigma a\alpha$.

4. The line joining $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

for which $l : m : n = Y-Z : Z-X : X-Y$ with $X = x_1 - x_2$, etc.

If pq be the projection of PQ on OD,

$$PQ = (Qq - Pp)/\cos\theta_1 = (Y-Z)/2a\cos\theta_1;$$

$$\therefore\ 16\Delta^2 PQ^2 = \Sigma a^2(X-Y)(X-Z)$$
$$= \Sigma a^2X^2 - \Sigma 2bcYZ\cos A.$$

The area of the triangle PQR, where R is (x_3, y_3, z_3),

$$= \frac{1}{4}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \times \frac{PQ}{\sqrt{-\Sigma a^2(Z-X)(X-Y)}}$$

$$= \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} / 16\Delta.$$

The distance of a point $P(x, y, z)$ from O is given by

$$16\Delta^2 . OP^2 = \Sigma a^2x^2 - \Sigma 2bcyz\cos A,$$

while a point circle at O is

$$\Sigma ax\cos A = \tfrac{1}{2}abc,$$

and therefore $OP^2 = (\Sigma ax\cos A - \frac{1}{2}abc)/\Sigma a\cos A$;

on equating these two expressions for OP^2 we get the identical relation of § 1.

5. The radical axis of the circles $\Sigma lx = s$ and $\Sigma l'x = s'$ is $\Sigma(l - \kappa l')x = s - \kappa s'$ where κ is given by $\Sigma l = \kappa\Sigma l'$.

A point circle whose centre has its trilinear coordinates α, β, γ is

$$\Sigma aax = 2R\Sigma a\beta\gamma ;$$

therefore, if α, β, γ is on the circle $\Sigma lx = s$, the tangent at α, β, γ is

$$\Sigma(2\Delta l - a\alpha\Sigma l)x = 2\Delta s - 2R\Sigma l \,.\, \Sigma a\beta\gamma.$$

Examples.

(i) The point of intersection of the lines $lx+my+nz=s$ and $\Sigma lx'=s$, $\Sigma l=\Sigma l'=0$, is

$$x = R^2 - \Sigma a^2 MN/16\Delta^2 T^2 + R(bN\cos B - cM\cos C), \text{ etc.},$$

where $M = sl' - s'l$, etc., and $T = lm_1 - l_1 m$.

(ii) If P be the point (x, y, z) and Q the point (α, β, γ), then

$$2\Delta \,.\, PQ^2 = \Sigma aax - 2R\Sigma a\beta\gamma.$$

(iii) The circle on KG as diameter is

$$\Sigma(4a^2+b^2+c^2)x = 8\Delta^2 \operatorname{cosec}^2\omega,$$

and if t_a, etc., denote the tangents to this circle from A, B, C, $\Sigma t_a^2 = \Sigma a^2/3$.

The Equation of the Second Degree.

6. The general equation of the second degree in tripolars may be written

$$lx^2+my^2+nz^2+l'yz+m'zx+n'xy+ix+jy+kz+s=0.$$

Converting to Cartesians by means of the formulae of § 1, this becomes

$$(l+m+n+l'+m'+n')(\xi^2+\eta^2)^2+(P\xi+Q\eta)(\xi^2+\eta^2)+\ldots=0,$$

in general a bicircular quartic, reducing to a circular cubic if

$$l+m+n+l'+m'+n'=0,$$

and to a conic if, in addition,

$$P \equiv -4cl\cos B - 4an - 2al' - (2a + c\cos B)m' - 2cn'\cos B = 0,$$
$$Q \equiv -2c\sin B(2l + m' + n') = 0,$$

giving as conditions for a conic

$$\left.\begin{aligned} l' &= l-m-n \\ m' &= m-l-n \\ n' &= n-m-l \end{aligned}\right\}.$$

Now the coefficients of the terms of the second degree in the identical relation also satisfy these conditions, and therefore the equation of a conic can always be written in the form

$$\Sigma l(x-y)(x-z)+ix+jy+kz+s=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

or, by adding p times the general relation where $p=\Sigma i/16\Delta^2$, in the form

$$\Sigma l(x-y)(x-z)+ix+jy+kz+s=0, \quad \Sigma i=0. \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

In this form the condition that the equation represents (a) two straight lines is $s(l^2+m^2+n^2-2mn-2nl-2lm) = ljk+mik+nji$;

(b) a circle, $l : m : n = a^2 : b^2 : c^2$;

(c) a parabola, $\Sigma l^2 = 2\Sigma lm$;

(d) a rectangular hyperbola, $\Sigma bc\cos A \,.\, l = 0$.

These conditions become more complicated for the form (i).

7. From the results of the last section it follows that, if we transform the ordinary trilinear equation of a conic into tripolars, we need only calculate the coefficients of x^2 and x besides the absolute term. Proceeding in this way, the conic $ua^2+\ldots+2u'\beta\gamma+\ldots=0$ appears immediately in the 'normal

or $\Sigma i=0$ form; the coefficients are, however, far from simple, but on subtracting the resulting equation from

$$u+v+w-2u'\cos A-2v'\cos B-2w'\cos C$$

times the identical relation, the equation becomes

$$\begin{aligned}\Sigma a^2(vc^2+wb^2-2u'bc)(x-y)(x-z)\\ -2\Sigma a^2bcx\{bc(v+w)+u'(2a^2-b^2-c^2)-abv'-acw'\}\\ +a^2b^2c^2\{a^2u+b^2v+c^2w-2u'bc-2v'ca-2w'ab\}=0.\end{aligned}$$

Thus the circumconic $\Sigma l\beta\gamma=0$ becomes

$$\Sigma al(x-y)(x-z)+\Sigma ax(l\sqrt{2a^2-b^2-c^2}-abm-acn)+abc\Sigma bcl=0.$$

This may also be written in the form

$$\Sigma al\{(x-y+c^2)(x-z+b^2)-4bcx\cos A\}=0.$$

The inconic $\Sigma\sqrt{la}=0$ becomes

$$\begin{aligned}\Sigma a^2(cm+bn)^2(x-y)(x-z)-2\Sigma a^2bcx\{(la+mb+nc)(mc+nb)-2a^2mn\}\\ +a^2b^2c^2(\Sigma al)^2=0.\end{aligned}$$

The conic $\Sigma ua^2=0$ becomes

$$\Sigma a^2(vc^2+wb^2)(x-y)(x-z)-2\Sigma a^2b^2c^2(v+w)x+a^2b^2c^2\Sigma a^2u=0.$$

G. N. BATES.

MATHEMATICAL NOTES.

108. [D. 6. d.] *Two Minor Complaints.*

1. If at any time it becomes customary to conclude every example of integration with a simple numerical application, the hyperbolic functions will soon come into common use.

Thus $\log_e\dfrac{x+\sqrt{a^2-x^2}}{a}$ and the function usually written $\sinh^{-1}\dfrac{x}{a}$ are synonymous. It will not take long to try which expression is the more easily evaluated when (say) $x=\cdot37a$. Several writers have however expressed dissatisfaction at the existing notation. The notation current in most English text-books—$\sinh x$, $\cosh x$, $\tanh x$—is unpronounceable or worse.

Professor Minchin has recently testified against the current notation and suggested hysin x, hycos x and so on. Professors Merriman and Woodward, *Higher Mathematics*, employ the current notation except that they write

csch for cosech.

Professor Greenhill writes

$$\mathrm{sh}\,(x),\quad \mathrm{ch}\,(x),\quad \mathrm{th}\,(x),$$

by analogy with the accepted

$$\mathrm{sn}\,(u),\quad \mathrm{cn}\,(u),\quad \mathrm{dn}\,(u),$$

of the elliptic functions.

The symbols are of course read letter by letter $\mathrm{s}-\mathrm{h}-x$.

It is not quite obvious how $\coth x$, $\mathrm{sech}\,x$ and $\mathrm{cosech}\,x$ would be dealt with on this plan.

It is submitted (1) that some modification of the current notation is seriously needed; (2) that the Mathematical Association might be able to recognise a standard notation and procure its general adoption.

[Houels, *Recueil de Formules*, 2te. 50, contains a very compact four-figure table of hyperbolic functions.]

2. [X. 2.] It is much to be desired that some authoritative expression of opinion could be vouchsafed as to the use or disuse of Tabular logarithms.

The invariable practice of professional computers is strong evidence in favour of the existing tables.

Further, the printing of negative characteristics will always involve difficulty, additional expense, and risk of error; though of course if once a correct set of 'stereo' plates are made, the expense and risk of error have been overcome.

On the other hand the arbitrary 10 certainly causes difficulty to beginners. It also is apt to lead to carelessness as regards characteristics, whence frequently $L\sin\theta=9{\cdot}7$ when it should be $L\sin\theta=8{\cdot}7$.

The use of logarithm tables is certainly one of those matters where uniformity in practice is desirable. To work out an example before a class with the certainty that the plan one adopts (or better $\mathrm{L}\sin\theta=9{\cdot}3$ or $\log\sin\theta=\bar{1}{\cdot}3$) will be resented by half the audience—is a little trying. C. S. J.

[In connection with this Note, Mr. E. T. Whittaker suggests that it is not advisable to make changes in mathematical notation in this country unless there is a reasonable expectation that our practice will be followed on the continent. We suffer already from neglect of this principle.

What we most need at present in the region of function-notation is to replace the English $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, by the continental arcsin, arccos, arctan.]

109. [D. 6. d.] *Mnemonic for hyperbolic formulae.*

Hyperbolic functions are now so constantly used, that a brief mnemonic for their somewhat confusing formulae may not be unwelcome.

In any Trigonometrical formula for θ, 2θ, 3θ, or θ and ϕ, after changing *sin* to *sinh*, *cos* to *cosh*, etc., change the sign of any term that contains (or implies) a *product* of *sinhs*, *e.g.* $\tanh\theta\tanh\phi$ implies a product of sinhs,

$$\therefore\ \tanh(\theta+\phi)=\frac{\tanh\theta+\tanh\phi}{1+\tanh\theta\tanh\phi}$$

$$\sinh 3\theta=3\sinh\theta+4\sinh 3\theta\,;$$

$$\cosh\theta-\cosh\phi=+2\sinh\frac{\theta+\phi}{2}\sinh\frac{\theta-\phi}{2}\,;$$

and so on. This rule would fail for terms of the 4th degree, but it covers everything that is likely to be required, and is very convenient for teaching purposes. G. OSBORN.

110. [K. 6. a.] *To find the distance between two points, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ in trilinear coordinates.*

Let r=distance, θ, ϕ, ψ the direction angles of the straight line joining the points.

$$\therefore\ \frac{\cos\theta}{\alpha_1-\alpha_2}=\frac{\cos\phi}{\beta_1-\beta_2}=\frac{\cos\psi}{\gamma_1-\gamma_2}=\frac{1}{r}\ \text{also}\ \phi-\theta=\pi-C,\ \text{etc.}$$

$$\text{Now } 4\sin(\phi-\theta)\cos\theta\cos\phi=2\sin(\phi-\theta)\cos(\theta+\phi)+2\sin(\phi-\theta)\cos(\phi-\theta),$$

$$=\sin 2\phi-\sin 2\theta+\sin 2(\phi-\theta)\,;$$

$$\therefore\ 4\Sigma\sin C\cos\theta\cos\phi=\Sigma\sin 2A=4\sin A\sin B\sin C\,;$$

$$\therefore\ \frac{1}{r^2}\Sigma\sin C(\alpha_1-\alpha_2)(\beta_1-\beta_2)=\sin A\sin B\sin C,$$

or
$$r^2=\Sigma\sin C(\alpha_1-\alpha_2)(\beta_1-\beta_2)/\sin A\sin B\sin C.$$

[Done in Pupil Room by Knox, ma., K. S., Eton College.]

111. [A. 1. c.] *Note on the Multinomial Theorem.*

Let $U=1+c_1x+c_2x^2+c_3x^3+\ldots$.

The expansion of U^n in a series of ascending powers of x is not to be found, as far as I know, in text-books of Algebra; but it is not difficult to establish a relation between the coefficients, and this method gives, I think, more definite grip to a theorem that usually strikes a beginner as somewhat ineffective.

As it is necessary to avoid using the notation of the Differential Calculus the method is given in full.

Let $U^n = 1 + k_1 x + k_2 x^2 + k_3 x^3 + \ldots$.

(For questions of convergence, see Chrystal's *Algebra*.)

Change x to $x+h$, and let

$$V = 1 + c_1(x+h) + c_2(x+h)^2 + \ldots$$
$$= U + hX_1,$$

where $X_1 = c_1 + 2c_2 x + 3c_3 x^2 + \ldots +$ terms in h, h^2, etc.

Similarly
$$V^n = 1 + k_1(x+h) + k_2(x+h)^2 + \ldots$$
$$= U^n + hX_2,$$

where $X_2 = k_1 + 2k_2 x + 3k_3 x^2 + \ldots +$ terms in h, etc.

Thus $V^{n+1} = V^n \, V = (U^n + hX_2)(U + hX_1)$

$\therefore$ $V^{n+1} - U^{n+1} = h(U^n X_1 + UX_2) + h^2 X_1 X_2$.

Now the left side is

$$(V - U)(V^n + V^{n-1} U + \ldots + U^n);$$

the first factor is hX_1, and on substituting and dividing the equation by h, we get $X_1(V^n + V^{n-1} U + \ldots + U^n) = U^n X_1 + UX_2 + h \, . \, X_1 X_2$.

Put $h=0$, so that V becomes U;

the expression $V^n + \ldots + U^n$ contains $\overline{n+1}$ terms, and becomes $(n+1)\,U^n$;

$X_1 + X_2$ lose all their terms in h, and become, say, $\xi_1 + \xi_2$;

the equation reduces to

$$\xi_1 (n+1)\, U^n = U^n \xi_1 + U\xi_2,$$

or
$$n\xi_1 \, U^n = \xi_2 \, U;$$

that is,
$$n(c_1 + 2c_2 x + 3c_3 x^2 + \ldots)(1 + k_1 x + k_2 x^2 + \ldots)$$
$$= (k_1 + 2k_2 x + 3k_3 x^2 + \ldots)(1 + c_1 x + c_2 x^2 + \ldots).$$

Picking out the coefficients of x^r on each side, we obtain

$$n(c_1 k_r + 2c_2 k_{r-1} + \ldots + rc_r k_1 + \overline{r+1}\, c_{r+1})$$
$$= \overline{r+1}\, k_{r+1} + rc_1 k_r + \overline{r-1}\, c_2 k_{r-1} + \ldots + 2c_{r-1} k_2 + c_r k_1 :$$

or $(r+1)k_{r+1} + (r-n)c_1 k_r + (r-1-2n)c_2 k_{r-1} + (r-2-3n)c_3 k_{r-2} + \ldots = 0$,

where terms containing negative suffixes are to be rejected, and we put $k_0 = 1$.

In practice the number of the c's is *small*, and starting with $r=0$, we can use the above relation to calculate several of the k's successively, with perhaps not more trouble than is required, by the usual method, for the last one alone.

Two cases should perhaps be mentioned specially.

The expansion of U^{-1} is of course a recurring series, and on putting $n=-1$ in the formula, we get

$$(r+1)(k_{r+1} + c_1 k_r + c_2 k_{r-1} + \ldots + c_{r+1}) = 0,$$

giving a scale of relation, for the k's, with constant coefficients.

If $n=1$, the expansion merely repeats U, and accordingly the relation

$$(r+1)k_{r+1} + (r-1)c_1 k_r + (r-3)c_2 k_{r-1} + \ldots = 0$$

gives in every case $k_i = c_i$;

hence, when there is only a finite number of c's, this relation will be found to give 0 for every k beyond that number (starting as before with $k_0 = 1$, $r=0$).

[As an example of the calculations, show that

$$(1 + x + x^2 - x^3)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{1}{16}x^3 + \ldots$$

which of course can easily be verified.]

G. OSBORN.

112. [K. 20. e.] A further consideration of the problem 391. [K. 20. e.] by M. E. N. Barisien may prove to be of interest. A solution was given by Mr. C. E. Youngman in the previous issue (p. 180).

"In any triangle prove that

$$(a+b-2c)^2\sec^2\frac{C}{2}+(a-b)^2\operatorname{cosec}^2\frac{C}{2}=\text{two similar expressions},$$

and interpret geometrically."

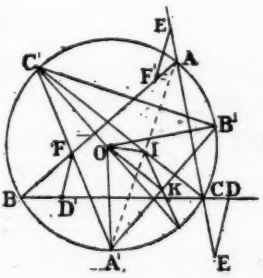
The first expression

$$\begin{aligned}
&= 4R^2\left\{(\sin A+\sin B-2\sin C)^2\sec^2\frac{C}{2}+(\sin A-\sin B)^2\operatorname{cosec}^2\frac{C}{2}\right\}\\
&= 4R^2\left\{4\left(\cos\frac{A-B}{2}-2\sin\frac{C}{2}\right)^2+4\sin^2\frac{A-B}{2}\right\}\\
&=16R^2\left\{1-4\cos\frac{A-B}{2}\cos\frac{A+B}{2}+2(1-\cos C)\right\}\\
&=16R^2\{3-2(\cos A+\cos B+\cos C)\}\\
&=16R^2\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\\
&=16(R^2-2Rr)=16OI^2,
\end{aligned}$$

where O is the circumcentre and I the incentre.

Obviously the other two expressions are equal to this.

As suggested by Mr. Youngman in the last issue, on BC, CA, AB take $BD=c$, $CE=a$, $AF=b$; and again on CB, BA, AC take $CD'=b$, $BF'=a$, $AE'=c$. Then $\overline{DE'}=\overline{DB}+\overline{BA}+\overline{AE'}$ = sum of three vectors whose directions are along the sides of the triangle taken in order, and *whose lengths are each equal to c.*



I propose to show that $DE'=\frac{c}{R}.OI$ and is perpendicular to it.

Let the perpendiculars from O to the sides of the triangle meet the circumcircle in A', B', C' respectively.

Then it can be shown that I is the orthocentre of the $\triangle A'B'C'$.

Hence if OK be drawn perpendicular to $A'B'$ and bisecting it, we have

$$\overline{C'I}=2\overline{OK}=\overline{OA'}+\overline{OB'};$$

$$\therefore\ \overline{OA'}+\overline{OB'}+\overline{OC'}=\overline{OI}.$$

Hence three vectors along BC, CA, AB, each of length R add up to OI, perpendicular to OI;

$$\therefore\ DE'=\frac{c}{R}.OI, \text{ and is perpendicular to } OI.$$

It immediately follows that

$$EF'=\frac{a}{R}OI,\quad FD'=\frac{b}{R}OI;$$

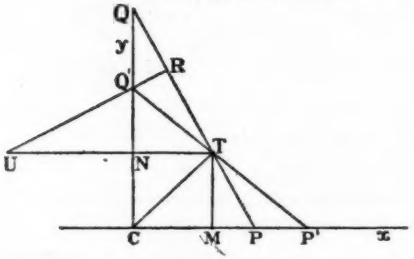
$$\therefore\ DE':EF':FD'=c:a:b,$$

and the three are parallel to each other, each being perpendicular to OI.

Finally, since $DE'=\frac{c}{R}OI=2OI\sin C$, the circumradius of the $\triangle DE'C$ is OI; as is also the case with the two other triangles. F. Glanville Taylor.

113. [L¹. 11. a.] *Let Cx, Cy be the asymptotes of a R^a. H^a. PQ, PQ any two tangents (intersecting in T), so that $CP.CQ=CP'.CQ'$. (PQ, $P'Q'$ are parallel.)*

Since
$$\triangle CPQ=\triangle CP'Q',$$
$$\triangle PTP'=\triangle QTQ',$$
or
$$PP'.TM=QQ'.TN=QQ'.CM;$$
$$\therefore \frac{TM}{CM}=\frac{QQ'}{PP'}=\frac{CQ}{CP},$$
$$T\hat{C}M=C\hat{P}Q \text{ (or } C\hat{P}'Q').$$



Let U be the orthocentre of the $\triangle QTQ'$.

Then
$$T\hat{C}M=U\hat{T}C,$$
and
$$C\hat{P}Q=C\hat{R}Q', \text{ since } C, P, R, Q' \text{ are concyclic,}$$
$$=C\hat{R}U,$$
$$U\hat{T}C=C\hat{R}U, \text{ and } C, T, R, U \text{ are concyclic;}$$
$$\therefore UCT \text{ is a right angle,}$$
and
$$UC^2=UN.UT,$$
so that C lies on the polar circle of $\triangle QTQ'$.

Taking any three tangents with the asymptote Cy:

The polar circles of the three triangles formed by taking any two tangents with the asymptote will (by the above) all pass through C; hence the polar circle of the triangle formed by the three tangents will also pass through C. For the four polar circles are coaxal (see Nixon or M'Clelland). R. F. Davis.

SOLUTIONS.

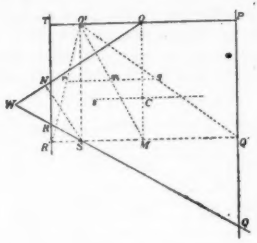
392. [L¹. 4. c.] *Two parallel tangents TR, PQ are drawn to a conic; O is the mid-point of TP, a tangent perpendicular to the former two, and RSQ is perpendicular to SO. If W be the fourth harmonic to R, S, Q, then is WO a tangent to the conic.* C. V. Durell.

Solution by R. F. Davis.

This property can be obtained by reciprocation. Let ABC be a triangle; D the middle point of BC, and AX the perpendicular from A on BC. Take P so that $\{PBXC\}=-1$; then $PD.PX=PB.PC=PA.PQ$ if PA meet the circumcircle of ABC again in Q. Thus $ADXQ$ is concyclic; and the angle $DQA=AXD=$a right angle. Then reciprocate with respect to X.

Solution by C. E. YOUNGMAN.

Let $O'Q'R'N$ be the projections of S on TO, PQ, TR, and OW; we have to prove that N lies on the circle $O'Q'R'$, centre C. Bisect $Q'R'$ at M, and $O'Q'$, $O'R'$, $O'M$ at q, r, m; and draw Cs parallel to qmr. Then the angle OMS is right, and $SNO'OM$ lie on a circle whose centre is m; $\therefore \angle SO'N = SON$



And the figures $O'Q'SR'$ and $OQSR$ are evidently similar; $\therefore$ the pencil $O'(NR'SQ')$ is harmonic, like $O(NRSQ)$. So is $C(mrsq)$, which has its last three rays perpendicular to those of $O'(NR'SQ')$; $\therefore$ Cm is perp. to $O'N$; whence $CN = CO'$.

394. [P. 2. a.] *A variable conic S touches a given line at a given point P and is reciprocated with respect to any fixed origin O into a conic Σ touching a given line at a given point π: prove that the product of the radii of curvature of S at P and Σ at π is constant for all pairs of conics such as S, Σ.* A. P. THOMPSON.

Solution by R. F. DAVIS.

Generally for any two reciprocal curves $rP = pR = \kappa^2$. This leads to $(r\,dr/dp).(R\,dR/dP) = r^3R/p^2 = \text{constant}$ in case under consideration.

Solution by C. V. DURELL.

Draw a conic focus O and having contact of the second order with S at P. Let C be its centre, $CA.CB$ its axes, and CD the semi-diameter conjugate to CP: draw OY perpendicular to the tangent at P. Then this conic reciprocates into the circle of curvature.

$$\therefore \text{ new radius of curvature} = \kappa^2 \cdot \frac{CA}{CB^2} = \rho_1.$$

But conic centre C has same circle of curvature as S;

$$\therefore \text{ radius of curvature of } S = \frac{CD^3}{CA.CB} = \rho;$$

$$\therefore \rho.\rho_1 = \kappa^2 \cdot \frac{CD^3}{CB^3} = \kappa^2 \cdot \frac{OP^3}{OY^3}$$

$$= \text{constant.} \qquad \text{Q.E.D.}$$

396. [I. 25. b.] *Required to find a triangular number which can be expressed as the sum of two other triangular numbers in five different ways.* [*N.B.—There is only one solution in numbers less than* 5000.] W. ALLEN WHITWORTH.

Solution by J. BLAIKIE.

It is required to find five sets of values of p and q which, along with the same value of n (less than 100), will satisfy the equation

$$\tfrac{1}{2}n(n-1) = \tfrac{1}{2}p(p-1) + \tfrac{1}{2}q(q-1) \ldots\ldots\ldots\ldots (1).$$

This equation may be put into the form

$$q(q-1)=(n-p)(n+p-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Thus if the product of $q(q-1)$ can be expressed as a product of two other factors, of which one is even and the other odd, we can find values of n and p which along with the assumed value of q will satisfy equation (1). For example, if $q=13$, we have $q(q-1)=1\times156=3\times52=$etc. These lead to $n=79$, $p=78$; $n=28$, $p=25$; etc. By performing the same process with other values of q from 13 to 55, we find that in five cases n is 79, giving 3081 as the triangular number; the five pairs of other triangular numbers whose sum is equal to 3081 being given by the following table:

p	q	$\frac{1}{2}p(p-1)$	$\frac{1}{2}q(q-1)$
78	13	3003	78
76	22	2850	231
70	37	2415	666
63	48	1953	1128
57	55	1596	1485

[The next solution is 10731 given by $n=147$, while for $n=154$ there are *seven* sets of values for p, q.]

399. [L¹. 17. e. 19. d.] *Two parabolas have a common focus; from any point on a common tangent are drawn the other tangents to the parabolas; shew that another parabola with the same focus can be drawn touching the last two lines and the join of their points of contact.* (C.)

Solution by R. F. Davis and C. V. Durell.

This property can be obtained by reciprocation. Let two circles intersect in R, S, and through R draw any chord URV. Also let the tangents at U, V intersect in T. Then the circumcircle of UVT passes through S.

Reciprocating with respect to S, etc.

Solution by C. V. Durell and C. E. Youngman.

Let PQ, PQ' be the two parabolas, S the focus, PTP' a common tangent, TQ, TQ' the other tangents. Then STP, SQT are similar triangles; also STP', $SQ'T$; therefore the angles SQT, $SQ'T$ are supplementary or equal according as T does or does not lie between P and P'. At the same time ST does or does not pass between Q and Q'; therefore always $STQQ'$ are concyclic, and a parabola with focus S can be inscribed to TQQ'.

409. [M¹. 3. j.] *Lines are drawn from every point of a curve making an angle β with the tangent at the point: if $(\bar{x}, \bar{y})$ be the centre of curvature at a point (x, y), and (ξ, η) the point corresponding to (x, y) on the envelope of the above lines, find ξ and η in terms of x, y, $\bar{x}$, $\bar{y}$, and β.* (C.)

Solution by W. F. Beard and J. F. Hudson.

Let P, Q be two consecutive points on the curve, C the centre of curvature, and R the corresponding point on the envelope of a line making an angle β with PQ.

Then $$C\hat{P}R = C\hat{Q}R = 90° - \beta;$$

$\therefore$ the circle on CP as diameter passes through R, and $PR = \rho \sin\beta$.

Equation of PR is $$\frac{\xi - x}{\cos(\beta+\psi)} = \frac{\eta - y}{\sin(\beta+\psi)} = \rho \sin\beta,$$

but $$x = \bar{x} + \rho\sin\psi,$$
$$y = \bar{y} - \rho\cos\psi;$$
$$\therefore\ \xi = x + \sin\beta\cos\beta\,.\,(\bar{y} - y) - \sin^2\beta\,.\,(x - \bar{x}),$$
$$\eta = y + \sin^2\beta\,.\,(y - y) + \sin\beta\cos\beta(x - \bar{x});$$

or $$\xi = x\cos^2\beta - y\sin\beta\cos\beta + \bar{x}\sin^2\beta + \bar{y}\sin\beta\cos\beta,$$
$$\eta = x\sin\beta\cos\beta + y\cos^2\beta - \bar{x}\sin\beta\cos\beta + \bar{y}\sin^2\beta.$$

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